Implicit Differentiation and the Chain Rule

The chain rule tells us that:

$$\frac{d}{dx}(f \circ g) = \frac{df}{dg}\frac{dg}{dx}.$$

While implicitly differentiating an expression like $x + y^2$ we use the chain rule as follows:

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy}\frac{dy}{dx} = 2yy'.$$

Why can we treat y as a function of x in this way?

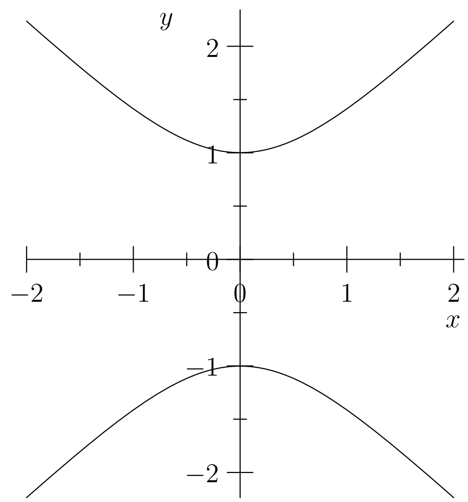


Figure 1: The hyperbola $y^2 - x^2 = 1$.

Consider the equation $y^2 - x^2 = 1$, which describes the hyperbola shown in Figure 1. We cannot write y as a function of x, but if we start with a point (x, y) on the graph and then change its x coordinate by sliding the point along the graph its y coordinate will be constrained to change as well. The change in y is *implied* by the change in x and the constraint $y^2 - x^2 = 1$. Thus, it makes sense to think about $y' = \frac{dy}{dx}$, the rate of change of y with respect to x.

Given that $y^2 - x^2 = 1$:

a) Use implicit differentiation to find y'.

b) Check your work by using Figure 1 to estimate the slope of the tangent line to the hyperbola when $y = -1$ and when $x = 1$.

c) Check your work for $y > 0$ by solving for y and using the direct method to take the derivative.

$y^2 - x^2 = 1$ 9/7/25

a) $2y\frac{dy}{dx} - 2x = 0$

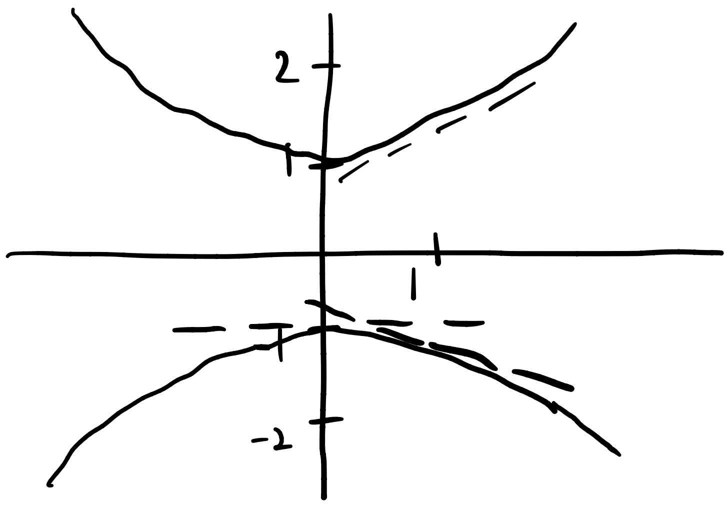
$y^2 = x^2 + 1$

$y = \pm\sqrt{x^2+1}$

$y\frac{dy}{dx} = x$

$\frac{dy}{dx} = \frac{x}{y} = \pm\frac{x}{\sqrt{x^2+1}}$

b)



When $y=-1$, $x=0$ and $\frac{dy}{dx}=0$.

When $x=1$, $y=\pm\sqrt{2}$ and $\frac{dy}{dx}=\frac{1}{\sqrt{2}}$ or $\frac{dy}{dx}=-\frac{1}{\sqrt{2}}$.

Both looks reasonable on the graph.

c) $y>0$, $y=\sqrt{x^2+1}$

$\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x$

$= \frac{x}{\sqrt{x^2+1}}$

$\therefore$ The direct method agrees with implicit differentiation.